Lecture 3

Probability - Part 1

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Outline

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Probability: Bayesian vs Frequentist Interpretations

- what is probability?
- there are actually at least two different interpretations of probability
  1. **frequentist**: probabilities represent long run frequencies of events (**trials**)
  2. **Bayesian**: probability is used to quantify our uncertainty about something (**information** rather than repeated trials)
- coin toss event:
  1. frequentist: if we flip the coin many times, we expect it to land heads about half the time
  2. Bayesian: we believe the coin is equally likely to land heads or tails on the next toss
- **advantage of the Bayesian** interpretation: it can be used to model our uncertainty about events that do not have long term frequencies; frequentist needs repetition
- **the basic rules of probability theory are the same**, no matter which interpretation is adopted

Outline

Foundations of Probability

In order to define a **probability space** we need 3 components $\{\Omega, \mathcal{F}, P\}$:

- **sample space** Ω: the set of all the outcomes of a random **experiment**. Here, each **outcome** (realization) $\omega \in \Omega$ can be thought of as a *complete description of the state of the real world* at the end of the experiment
- **event space** $\mathcal{F}$: a set whose elements $A \in \mathcal{F}$ (called **events**) are subsets of Ω (i.e., $A \subseteq \Omega$ is a collection of possible outcomes of an experiment)
  $\mathcal{F}$ should satisfy 3 properties (σ-algebra of events):
  1. $\emptyset \in \mathcal{F}$
  2. $A \in \mathcal{F} \Rightarrow \overline{A} = \Omega \setminus A \in \mathcal{F}$ (closure under complementation)
  3. $A_1, A_2, ... \in \mathcal{F} \Rightarrow \cup_i A_i \in \mathcal{F}$ (closure under countable union)
- **probability measure** P: a function $P : \mathcal{F} \rightarrow \mathbb{R}$ that satisfies the following 3 **axioms of probability**
  1. $P(A) \geq 0$ for all $A \in \mathcal{F}$
  2. $P(\Omega) = 1$
  3. if $A_1, A_2, ...$ are disjoint events (i.e., $A_i \cap A_j = \emptyset$ whenever $i \neq j$), then $P(\cup_i A_i) = \sum_i P(A_i)$ (P is countably additive)

A simple example

experiment: tossing a six-sided dice

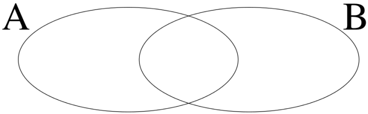
- sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ (a simple representation)
- trivial event space
  - $\mathcal{F} = \{\emptyset, \Omega\}$
  - unique probability measure satisfying the requirements is given by $P(\emptyset) = 0, P(\Omega) = 1$
- power set event space
  - $\mathcal{F} = 2^\Omega$ (i.e., the set of all subsets of Ω)
  - a possible probability measure $P(i) = 1/6$ for $i \in \{1, 2, 3, 4, 5, 6\} = \Omega$

question: do the above sample space outcomes completely describe the state of a dice-tossing experiment?

Probability Measure Properties

some **important properties** on events (can be inferred from axioms)

- $A \subseteq B \Rightarrow P(A) \leq P(B)$
- $P(A \cap B) \leq min(P(A), P(B))$
- union bound: $P(A \cup B) \leq P(A) + P(B)$
- complement rule: $P(\overline{A}) = P(\Omega \setminus A) = 1 - P(A)$
- impossible event: $P(\emptyset) = 0$
- law of total probability: if $A_1, ..., A_k$ are a set of disjoint events such that $\bigcup_{i=1}^{N} A_i = \Omega$ then $\sum_{i=1}^{N} P(A_i) = 1$



- general addition rule[1]: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

[1] events can be represented by using Venn diagrams

Conditional Probability

- let B be an event with non-zero probability, i.e. $p(B) > 0$
- the **conditional probability** of any event A given B is defined as

$$P(A|B) = \frac{p(A \cap B)}{p(B)}$$

- in other words, $P(A|B)$ is the probability measure of the event A *after observing the occurrence of event B*
- two **events** are called **independent** iff

$$P(A \cap B) = P(A)P(B) \ \ \text{(or equivalently } P(A|B) = P(A))$$

- therefore, **independence** is equivalent to saying that observing B does not have any effect on the probability of A

Conditional Probability

a frequentist intuition of conditional probability

- N is total number of experiment trials
- for an event E, let's define $P(E) \triangleq \frac{N_E}{N}$ where N_E is the number of trials where E is verified

hence for events A and B (considering the limit $N \rightarrow \infty$)

- $P(A) = \frac{N_A}{N}$ where N_A is the number of trials where A is verified
- $P(B) = \frac{N_B}{N}$ where N_B is the number of trials where B is verified
- $P(A \cap B) = \frac{N_{A \wedge B}}{N}$ where $N_{A \wedge B}$ is the number of trials where both A and B are verified

let's consider only the trials where B is verified, hence

- $P(A|B) = \frac{N_{A \wedge B}}{N_B}$ ($N_B > 0$ now acts as N)
- dividing by N, one obtains $P(A|B) = \frac{N_{A \wedge B}/N}{N_B/N} = \frac{P(A \cap B)}{P(B)}$

Outline

Random Variables

intuition: a random variable represents an interesting "aspect" of the outcomes $\omega \in \Omega$

more formally:

- a **random variable** X is a **function** $X : \Omega \rightarrow \mathbb{R}$
- a random variable is denoted by using **upper case letters** $X(\omega)$ or more simply X (here X is a function)
- the particular **values** (instances) of a random variable may take on are denoted by using **lower case letters** x (here $x \in \mathbb{R}$)

types of random variables:

- **discrete random variable**: function $X(\omega)$ can only take values in a finite set $\mathcal{X} = \{x_1, x_2, ..., x_m\}$ or countably infinite set (e.g. $\mathcal{X} = \mathbb{N}$)
- **continuous random variable**: function $X(\omega)$ can take continuous values in $\mathbb{R}$

Random Variables

a random variable is a measurable function

- since $X(\omega)$ takes values in $\mathbb{R}$, let's try to define an "*event space*" on $\mathbb{R}$: in general we would like to observe if $X(\omega) \in B$ for some subset $B \subset \mathbb{R}$
- as "event space" on $\mathbb{R}$, we can consider $\mathcal{B}$ the **Borel σ-algebra on the real line**[2], which is generated by the set of half-lines $\{(-\infty, a] : a \in (-\infty, \infty)\}$ by *repeatedly* applying union, intersection and complement operations
- an element $B \subset \mathbb{R}$ of the Borel σ-algebra $\mathcal{B}$ is called a **Borel set**
- the set of all open/closed subintervals in $\mathbb{R}$ are contained in $\mathcal{B}$
- for instance, $(a, b) \in \mathcal{B}$ and $[a, b] \in \mathcal{B}$
- a random variable is a **measurable function** $X : \Omega \to \mathbb{R}$, i.e.

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \quad \text{for each } B \in \mathcal{B}$$

  i.e., if we consider an "event" $B \in \mathcal{B}$ this can be represented by a proper event $F_B \in \mathcal{F}$ where we can apply the probability measure P

[2] here we should use the notation $\mathcal{B}(\mathbb{R})$, for simplicity we drop $\mathbb{R}$

Induced Probability Space

- we have defined the probability measure P on $\mathcal{F}$, i.e. $P : \mathcal{F} \to \mathbb{R}$
- how to define the probability measure P_X w.r.t. X?

$$P_X(B) \triangleq P(X^{-1}(B)) = P(\{\omega \in \Omega : X(\omega) \in B\})$$

  which is well-defined given that $X^{-1}(B) \in \mathcal{F}$
- at this point, we have an **induced probability space** $\{\Omega_X, \mathcal{F}_X, P_X\} \triangleq \{\mathbb{R}, \mathcal{B}, P_X\}$ and we can equivalently reason on it

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Discrete Random Variables

discrete Random Variable (RV)

- $X(\omega)$ can only take values in a finite set $\mathcal{X} = \{x_1, x_2, ..., x_m\}$ or in a countably infinite set
- how to define the probability measure P_X w.r.t. X?

$$P_X(X = x_k) \triangleq P(\{\omega : X(\omega) = x_k\})$$

- in this case P_X returns measure one to a countable set of reals
- a simpler way to represent the probability measure is to directly specify the probability of **each value** the discrete RV can assume
- in particular, a **Probability Mass Function** (PMF) is a function $p_X : \mathbb{R} \to \mathbb{R}$ such that

$$p_X(X = x) \triangleq P_X(X = x)$$

- it's very common to drop the subscript X and denote the PMF with $p(X) = p_X(X = x)$

Outline

Important Rules of Probability

considering two discrete RV X and Y at the same time

- **sum rule**

$$p(X) = \sum_Y p(X, Y) \qquad \text{(marginalization)}$$

- **product rule**

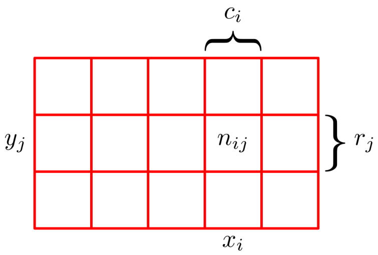
$$p(X, Y) = p(X|Y)p(Y)$$

- **chain rule**:

$$p(X_{1:D}) = p(X_1)p(X_2|X_1)p(X_3|X_2, X_1)...p(X_D|X_{1:D-1})$$

where $1 : D$ denotes the set $\{1, 2, ..., D\}$ (Matlab-like notation)

a frequentist intuition of the sum rule



- N number of trials
- n_{ij} number of trials in which $X = x_i$ and $Y = y_j$
- c_i number of trials in which $X = x_i$, one has $c_i = \sum_j n_{ij}$
- r_j number of trials in which $Y = y_j$, one has $r_j = \sum_i n_{ij}$
- $p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$ (considering the limit $N \to \infty$)

hence:

- $p(X = x_i) = \frac{c_i}{N} = \sum_j \frac{n_{ij}}{N} = \sum_j p(X = x_i, Y = y_j)$

Bayes' Theorem

combining the definition of condition probability with the product and sum rules:

1. $p(X|Y) = \frac{p(X,Y)}{P(Y)}$ (conditional prob. def.)
2. $p(X,Y) = p(Y|X)p(X)$ (product rule)
3. $p(Y) = \sum_X p(X,Y) = \sum_X p(Y|X)p(X)$ (sum rule + product rule)

one obtains the **Bayes' Theorem** (plug 2 e 3 into 1)

$$p(X|Y) = \frac{p(Y|X)p(X)}{\sum_X p(Y|X)p(X)}$$

N.B.: we could write $p(X|Y) \propto p(Y|X)p(X)$; the denominator $p(Y) = \sum_X p(Y|X)p(X)$ can be considered as a normalization constant

Bayes' Theorem

An Example

events:

- C = breast cancer present, $\overline{C}$ = no cancer
- M = positive mammogram test, $\overline{M}$ = negative mammogram test

probabilities:

- $p(C) = 0.4\%$ (hence $p(\overline{C}) = 1 - p(C) = 99.6\%$)
- if there is cancer, the probability of a pos mammogram is $p(M|C) = 80\%$
- if there is no cancer, we still have $p(M|\overline{C}) = 10\%$

false conclusion: positive mammogram $\Rightarrow$ the person is 80% likely to have cancer
question: what is the conditional probability $p(C|M)$?

$$p(C|M) = \frac{p(M|C)p(C)}{p(M)} = \frac{p(M|C)p(C)}{p(M|C)p(C) + p(M|\overline{C})p(\overline{C})}$$

$$= \frac{0.8 \times 0.004}{0.8 \times 0.004 + 0.1 \times 0.996} = 0.031$$

true conclusion: positive mammogram $\Rightarrow$ the person is about 3% likely to have cancer

Outline

Independence and Conditional Independence

considering two RV X and Y at the same time

- X and Y are **unconditionally independent**

$$X \perp Y \iff p(X,Y) = p(X)p(Y)$$

  in this case $p(X|Y) = p(X)$ and $p(Y|X) = p(Y)$

- $X_1, X_2, ..., X_D$ are **mutually independent** if

$$p(X_1, X_2, ..., X_D) = p(X_1)p(X_2)...p(X_D)$$

- X and Y are **conditionally independent**

$$X \perp Y|Z \iff p(X,Y|Z) = p(X|Z)p(Y|Z)$$

  in this case $p(X|Y,Z) = p(X|Z)$

Outline

continuous random variable

- $X(\omega)$ can take any value on $\mathbb{R}$
- how to define the probability measure P_X w.r.t. X?

$$P_X(X \in B) \triangleq P(X^{-1}(B)) \qquad \text{(with } B \in \mathcal{B}\text{)}$$

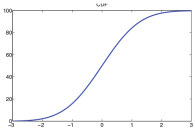
- in this case P_X gives zero measure to every singleton set, and hence to every countable set[3]

[3] unless we consider some particular/degenerate cases

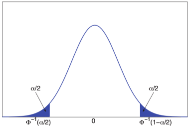
CDF and PDF
Definitions

given a continuous RV X

- **Cumulative Distribution Function** (CDF): $F(x) \triangleq P_X(X \leq x)$
  - $0 \leq F(x) \leq 1$
  - the CDF is a monotonically non-decreasing
    $F(x) \leq F(x + \Delta x)$ with $\Delta x > 0$
  - $F(-\infty) = 0,\ F(\infty) = 1$
  - $P_X(a < X \leq b) = F(b) - F(a)$



- **Probability Density Function** (PDF): $p(x) \triangleq \dfrac{dF}{dx}$
  - we assume F is continuous and the derivative exists
  - $F(x) = P_X(X \leq x) = \int_{-\infty}^{x} p(\xi) d\xi$
  - $P_X(x < X \leq x + dx) \approx p(x)dx$
  - $P_X(a < X \leq b) = \int_a^b p(x)dx$



$p(x)$ acts as a density in the above computations reconsider

1. $P_X(a < X \le b) = \int_a^b p(x)dx$
2. $P_X(a < X \le a + dx) \approx p(x)dx$

- the first implies $\int_{-\infty}^{\infty} p(x)dx = 1$ (consider $(a, b) = (-\infty, \infty)$)
- the second implies $p(x) \ge 0$ for all $x \in \mathbb{R}$
- it is possible that $p(x) > 1$, for instance, consider the **uniform distribution** with PDF

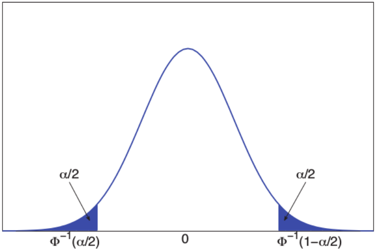
$$\text{Unif}(x|a, b) = \frac{1}{b-a}\mathbb{I}(a \le x \le b)$$

if $a = 0$ and $b = 1/2$ then $p(x) = 2$ in $[a, b]$

- assume F is continuous (this was required for defining $p(x)$)
- we have that $P_X(X = x) = 0$ (zero probability on a singleton set)
- in fact for $\epsilon \geq 0$:
  $P_X(X = x) \leq P_X(x - \epsilon < X \leq x) = F(x) - F(x - \epsilon) = \delta F(x, \epsilon)$
  and given that F is continuous $P_X(X = x) \leq \lim_{\epsilon \to 0} \delta F(x, \epsilon) = 0$

Quantile

- given that the CDF F is monotonically increasing, let's consider its inverse F^{-1}
- $F^{-1}(\alpha) = x_\alpha \iff P_X(X \leq x_\alpha) = \alpha$
- x_α is called the α **quantile** of F
- $F^{-1}(0.5)$ is the **median**
- $F^{-1}(0.25)$ and $F^{-1}(0.75)$ are the lower and upper **quartiles**
- for symmetric PDFs (e.g. $\mathcal{N}(0,1)$) we have $F^{-1}(1-\alpha/2) = -F^{-1}(\alpha/2)$ and the central interval $(F^{-1}(\alpha/2), F^{-1}(1-\alpha/2))$ contains $1-\alpha$ of the mass probability



Mean and Variance

- **mean or expected value** μ

for a discrete RV: $\mu = \mathbb{E}[X] \triangleq \sum_{x \in \mathcal{X}} x \; p(x)$

for a continuos RV: $\mu = \mathbb{E}[X] \triangleq \int_{x \in \mathcal{X}} x \; p(x) \; dx$ (defined if $\int_{x \in \mathcal{X}} |x| \; p(x) \; dx < \infty$)

- **variance** $\sigma^2 = \text{var}[X] \triangleq \mathbb{E}[(X - \mu)^2]$

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2] = \int_{x \in \mathcal{X}} (x - \mu)^2 p(x) dx =$$

$$= \int_{x \in \mathcal{X}} x^2 p(x) dx - 2\mu \int_{x \in \mathcal{X}} x p(x) dx + \mu^2 \int_{x \in \mathcal{X}} p(x) dx = \mathbb{E}[X^2] - \mu^2$$

(this can be also obtained for discrete RV)

- **standard deviation** $\sigma = \text{std}[X] = \sqrt{\text{var}[X]}$

Moments

- ***n*-th moment**

for a discrete RV: $\mathbb{E}[X^n] \triangleq \sum_{x \in \mathcal{X}} x^n \; p(x)$

for a continuos RV: $\mathbb{E}[X^n] \triangleq \int_{x \in \mathcal{X}} x^n \; p(x) \; dx$ (defined if $\int_{x \in \mathcal{X}} |x|^n \; p(x) \; dx < \infty$)

Outline

Binomial Distribution

- we toss a **coin** n times
- X is a discrete RV with $x \in \{0, 1, ..., n\}$, the occurred number of heads
- θ is the probability of heads
- $X \sim \text{Bin}(n, \theta)$ i.e., X has a **binomial distribution** with PMF

$$\text{Bin}(k|n, \theta) \triangleq \binom{n}{k} \theta^k (1-\theta)^{n-k} \qquad (= \ P_X(X = k))$$

where we use the **binomial coefficient**

$$\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$$

- mean $= n\theta$
- var $= n\theta(1-\theta)$

N.B.: recall that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Bernoulli Distribution

- we toss a **coin** only one time
- X is a discrete RV with $x \in \{0, 1\}$ where $1 =$ head, $0 =$ tail
- θ is the probability of heads
- $X \sim \text{Ber}(\theta)$ i.e., X has a **Bernoulli distribution** with PMF

$$\text{Ber}(x|\theta) \triangleq \theta^{\mathbb{I}(x=1)}(1-\theta)^{\mathbb{I}(x=0)} \qquad (= P_X(X = x))$$

that is

$$\text{Ber}(x|\theta) = \begin{cases} \theta & \text{if } x = 1 \\ 1-\theta & \text{if } x = 0 \end{cases}$$

- mean $= \theta$
- var $= \theta(1-\theta)$

Outline

Multinomial Distribution

- we toss a K-sided **dice** n times
- the possible outcome is $\mathbf{x} = (x_1, x_2, ..., x_K)$ where $x_j \in \{0, 1, ..., n\}$ is the number of times side j occurred
- $n = \sum_{j=1}^{K} x_j$
- θ_j is the probability of having side j
- $\sum_{j=1}^{K} \theta_j = 1$
- $X \sim \text{Mu}(n, \theta)$ i.e., X has a **multinomial distribution** with PMF

$$\text{Mu}(\mathbf{x}|n, \boldsymbol{\theta}) \triangleq \binom{n}{x_1 \; ... \; x_K} \prod_{j=1}^{K} \theta_j^{x_j}$$

where we use the **multinumial coefficient**

$$\binom{n}{x_1 \; ... \; x_K} \triangleq \frac{n!}{x_1! x_2! ... x_K!}$$

which is the num of ways to divide a set of size n into subsets of size $x_1, x_2, ..., x_K$

Multinoulli Distribution

- we toss the **dice** only one time
- the possible outcome is $\mathbf{x} = (\mathbb{I}(x_1 = 1), \mathbb{I}(x_2 = 1), ..., \mathbb{I}(x_K = 1))$ where $x_j \in \{0, 1\}$ represents if side j occurred or not (**dummy enconding** or **one-hot encoding**)
- θ_j is the probability of having side j, i.e., $p(x_j = 1|\boldsymbol{\theta}) = \theta_j$
- $X \sim \text{Cat}(\boldsymbol{\theta})$ i.e., X has the **categorical distribution** (or **multinoulli**)

$$\text{Cat}(\mathbf{x}|\boldsymbol{\theta}) = \text{Mu}(\mathbf{x}|1, \boldsymbol{\theta}) \triangleq \prod_{j=1}^{K} \theta_j^{x_j}$$

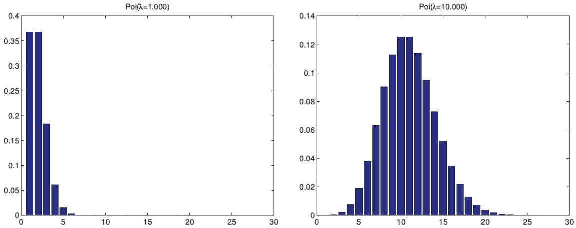
Outline

Poisson Distribution

- X is a discrete RV with $x \in \{0, 1, 2, ...\}$ (support on $\mathbb{N}^+$)
- $X \sim \text{Poi}(\lambda)$ i.e., X has a **Poisson distribution** with PMF

$$\text{Poi}(x|\lambda) \triangleq e^{-\lambda} \frac{\lambda^x}{x!}$$

- recall that $e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$
- this distribution is used as a model for counts of rare events (e.g. accidents, failures, etc)



Outline

Empirical Distribution

- given a dataset $\mathcal{D} = \{x_1, x_2, ..., x_N\}$
- the **empirical distribution** is defined as

$$p(x) = \sum_{i=1}^{N} w_i \delta_{x_i}(x)$$

- $0 \leq w_i \leq 1$ are the weights
- $\sum_{i=1}^{N} w_i = 1$
- $\delta_{x_i}(x) = \mathbb{I}(x = x_i)$
- this can be view as an **histogram** with "spikes" at $x_i \in \mathcal{D}$ and 0-probability out $\mathcal{D}$

Credits

- Kevin Murphy's book
- A. Maleki and T. Do *"Review of Probability Theory"*, Stanford University
- G. Chandalia *"A gentle introduction to Measure Theory"*