

Lecture 3

Probability - Part 2

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1 Common Continuous Distributions - Univariate

- Gaussian Distribution
- Degenerate PDFs
- Student's t Distribution
- Laplace Distribution
- Gamma Distribution
- Beta Distribution
- Pareto Distribution

2 Joint Probability Distributions - Multivariate

- Joint Probability Distributions
- Joint CDF and PDF
- Marginal PDF
- Conditional PDF and Independence
- Covariance and Correlation
- Correlation and Independence
- Common Multivariate Distributions

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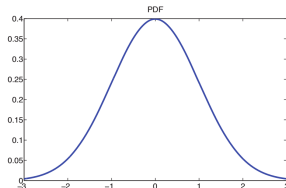
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Gaussian (Normal) Distribution

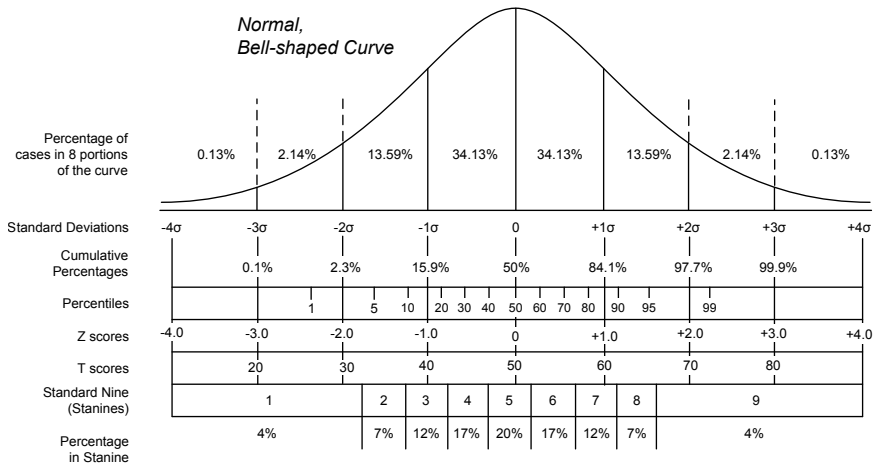
- X is a continuous RV with values $x \in \mathbb{R}$
- $X \sim \mathcal{N}(\mu, \sigma^2)$, i.e. X has a **Gaussian distribution** or **normal distribution**

$$\mathcal{N}(x|\mu, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad (= P_X(X=x))$$

- mean $\mathbb{E}[X] = \mu$
- mode μ
- variance $\text{var}[X] = \sigma^2$
- precision $\lambda = \frac{1}{\sigma^2}$
- $(\mu - 2\sigma, \mu + 2\sigma)$ is the approx 95% interval
- $(\mu - 3\sigma, \mu + 3\sigma)$ is the approx. 99.7% interval



Gaussian (Normal) Distribution



Gaussian (Normal) Distribution

why Gaussian distribution is the most widely used in statistics?

- 1 easy to interpret: just two parameters $\theta = (\mu, \sigma^2)$
- 2 **central limit theorem**: the sum of independent random variables has an approximately Gaussian distribution
- 3 least number of assumptions (maximum entropy) subject to constraints of having mean $= \mu$ and variance $= \sigma^2$
- 4 simple mathematical form, easy to manipulate and implement

homework: show that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = 1$$

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Degenerate PDFs

- X is a continuous RV with values $x \in \mathbb{R}$
- consider a Gaussian distribution with $\sigma^2 \rightarrow 0$

$$\delta(x - \mu) \triangleq \lim_{\sigma^2 \rightarrow 0} \mathcal{N}(x|\mu, \sigma^2)$$

- $\delta(x)$ is the **Dirac delta function**, with

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

- one has $\int_{-\infty}^{\infty} \delta(x) dx = 1$

$$\left(\lim_{\sigma^2 \rightarrow 0} \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) dx = 1 \right)$$

- **sifting property**

$$\int_{-\infty}^{\infty} f(x) \delta(x - \mu) dx = f(\mu)$$

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Student's t Distribution

- X is a continuous RV with values $x \in \mathbb{R}$
- $X \sim \mathcal{T}(\mu, \sigma^2, \nu)$, i.e. X has a **Student's t distribution**

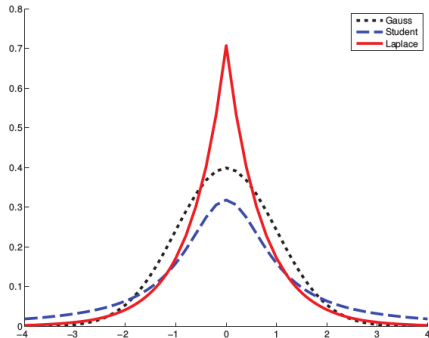
$$\mathcal{T}(x|\mu, \sigma^2, \nu) \propto \left[1 + \frac{1}{\nu} \left(\frac{x - \mu}{\sigma} \right)^2 \right]^{-\left(\frac{\nu+1}{2}\right)} \quad (= P_X(X = x))$$

- scale parameter $\sigma^2 > 0$
- degrees of freedom ν
- mean $\mathbb{E}[X] = \mu$ defined if $\nu > 1$
- mode μ
- variance $\text{var}[X] = \frac{\nu\sigma^2}{(\nu-2)}$ defined if $\nu > 2$

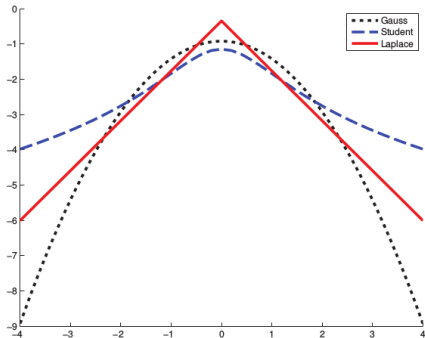
Student's t Distribution

- a comparison of $\mathcal{N}(0, 1)$, $\mathcal{T}(0, 1, 1)$ and $\text{Lap}(0, 1, \sqrt{2})$

PDF



$\log(\text{PDF})$



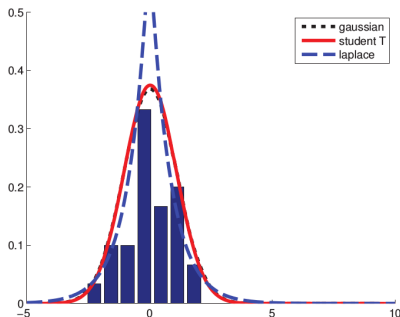
Student's t Distribution

Pros

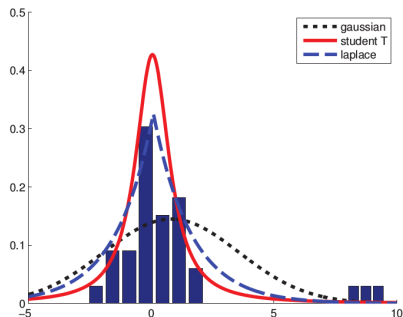
why should we use the Student distribution?

- it is less sensitive to outliers than the Gaussian distribution

without outliers



with outliers



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Laplace Distribution

- X is a continuous RV with values $x \in \mathbb{R}$
- $X \sim \text{Lap}(\mu, b)$, i.e. X has a **Laplace distribution**

$$\text{Lap}(x|\mu, b) \triangleq \frac{1}{2b} \exp\left(-\frac{|x - \mu|}{b}\right) \quad (= P_X(X = x))$$

- scale parameter $b > 0$
- mean $\mathbb{E}[X] = \mu$
- mode μ
- variance $\text{var}[X] = 2b^2$

compared to Gaussian distribution, Laplace distribution

- is more robust to outliers (see above)
- puts more probability density at μ (see above)

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Gamma Distribution

- X is a continuous RV with values $x \in \mathbb{R}^+$ ($x > 0$)
- $X \sim \text{Ga}(a, b)$, i.e. X has a **gamma distribution**

$$\text{Ga}(x|a, b) \triangleq \frac{b^a}{\Gamma(a)} x^{a-1} e^{-xb} \quad (= P_X(X = x))$$

- shape $a > 0$
- rate $b > 0$
- the **gamma function** is

$$\Gamma(x) \triangleq \int_{-\infty}^{\infty} u^{x-1} e^{-u} du$$

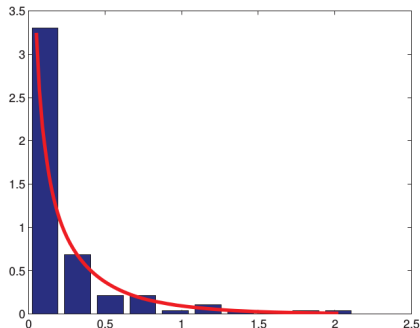
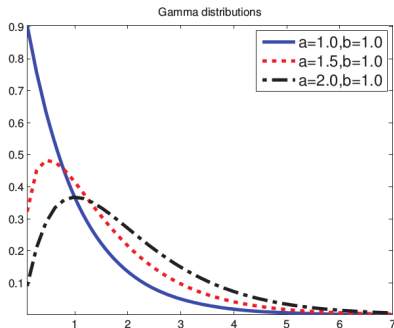
where $\Gamma(x) = (x-1)!$ for $x \in \mathbb{N}$ and $\Gamma(1) = 1$

- mean $\mathbb{E}[X] = \frac{a}{b}$
- mode $\frac{a-1}{b}$
- variance $\text{var}[X] = \frac{a}{b^2}$

N.B.: there are several distributions which are just special cases of the Gamma (e.g. exponential, Erlang, Chi-squared)

Gamma Distribution

- some $\text{Ga}(a, b = 1)$ distributions
- *right*: an empirical PDF of some rainfall data



Inverse Gamma Distribution

- X is a continuous RV with values $x \in \mathbb{R}^+$ ($x > 0$)
- if $X \sim \text{Ga}(a, b)$, i.e. $\frac{1}{X} \sim \text{IG}(a, b)$
- $\text{IG}(a, b)$ is the **inverse gamma distribution**

$$\text{IG}(x|a, b) \triangleq \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{-b/x} \quad (= P_X(X = x))$$

- mean $\mathbb{E}[X] = \frac{b}{a-1}$ (defined if $a > 1$)
- mode $\frac{b}{a+1}$
- variance $\text{var}[X] = \frac{b^2}{(a-1)^2(a-2)}$ (defined if $a > 2$)

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Beta Distribution

- X is a continuous RV with values $x \in [0, 1]$
- $X \sim \text{Beta}(a, b)$, i.e. X has a **beta distribution**

$$\text{Beta}(x|a, b) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} \quad (= P_X(X = x))$$

- requirements: $a > 0$ and $b > 0$
- the **beta function** is

$$B(a, b) \triangleq \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

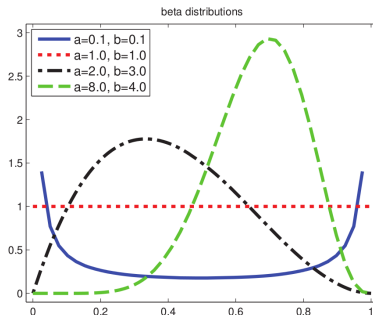
- mean $\mathbb{E}[X] = \frac{a}{a+b}$
- mode $\frac{a-1}{a+b-2}$
- variance $\text{var}[X] = \frac{ab}{(a+b)^2(a+b+1)}$

Beta Distribution

- **beta distribution**

$$\text{Beta}(x|a, b) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} \quad (= P_X(X = x))$$

- requirements: $a > 0$ and $b > 0$
- if $a = b = 1$ then $\text{Beta}(x|1, 1) = \text{Unif}(x|1, 1)$ in the interval $[0, 1] \subset \mathbb{R}$
- this distribution can be used to represent a prior on a probability value to be estimated



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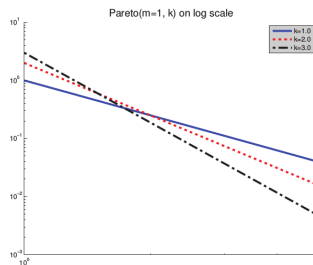
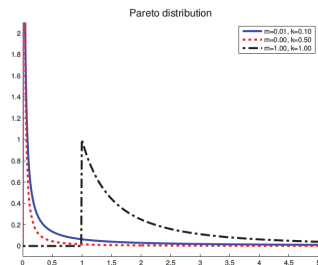
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Pareto Distribution

- X is a continuous RV with values $x \in \mathbb{R}^+$ ($x > 0$)
- $X \sim \text{Pareto}(k, m)$, i.e. X has a **Pareto distribution**

$$\text{Pareto}(x|k, m) = km^k x^{-(k+1)} \mathbb{I}(x \geq m) \quad (= P_X(X = x))$$

- as $k \rightarrow \infty$ the distribution approaches $\delta(x)$
- mean $\mathbb{E}[X] = \frac{km}{k-1}$ (defined for $k > 1$)
- mode m
- variance $\text{var}[X] = \frac{m^2 k}{(k-1)^2(k-2)}$
- this distribution is particularly useful for its **long tail**



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Joint Probability Distributions

- consider an ensemble of RVs X_1, \dots, X_D
- we can define a new RV $\mathbf{X} \triangleq (X_1, \dots, X_D)^T$
- we are now interested in modeling the stochastic relationship between X_1, \dots, X_D
- in this case $\mathbf{x} = (x_1, \dots, x_D)^T \in \mathbb{R}^D$ denotes a particular value (instance) of \mathbf{X}

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Joint Cumulative Distribution Function

Definition

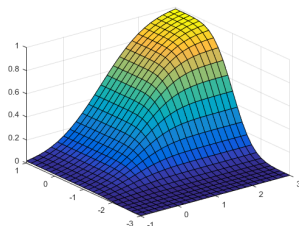
given a continuous RV \mathbf{X} with values $\mathbf{x} \in \mathbb{R}^D$

- **Cumulative Distribution Function (CDF)**

$$F(\mathbf{x}) = F(x_1, \dots, x_D) \triangleq P_{\mathbf{X}}(\mathbf{X} \leq \mathbf{x}) = P_{\mathbf{X}}(X_1 \leq x_1, \dots, X_D \leq x_D)$$

- properties:

- $0 \leq F(\mathbf{x}) \leq 1$
- $F(x_1, \dots, x_j, \dots, x_D) \leq F(x_1, \dots, x_j + \Delta x_j, \dots, x_D)$ with $\Delta x_j > 0$
- $\lim_{\Delta x_j \rightarrow 0^+} F(x_1, \dots, x_j + \Delta x_j, \dots, x_D) = F(x_1, \dots, x_j, \dots, x_D)$ (right-continuity)
- $F(-\infty, \dots, -\infty) = 0$
- $F(+\infty, \dots, +\infty) = 1$



Joint Probability Density Function

Definitions

given a continuous RV \mathbf{X} with values $\mathbf{x} \in \mathbb{R}^D$

- **Probability Density Function (PDF)**

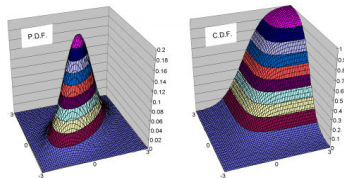
$$p(\mathbf{x}) = p(x_1, \dots, x_D) \triangleq \frac{\partial^D F}{\partial x_1, \dots, \partial x_D}$$

we assume the above partial derivative of F exists

- properties:

- $F(\mathbf{x}) = P_{\mathbf{X}}(\mathbf{X} \leq \mathbf{x}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_D} p(\xi_1, \dots, \xi_D) d\xi_1 \dots d\xi_D$
- $P_{\mathbf{X}}(\mathbf{x} < \mathbf{X} \leq \mathbf{x} + d\mathbf{x}) \approx p(\mathbf{x}) dx_1 \dots dx_D = p(\mathbf{x}) d\mathbf{x}$
- $P_{\mathbf{X}}(\mathbf{a} < \mathbf{X} \leq \mathbf{b}) = \int_{a_1}^{b_1} \dots \int_{a_D}^{b_D} p(\mathbf{x}) dx_1 \dots dx_D$

N.B.: $p(\mathbf{x})$ acts as a density in the above computations



Joint Probability Density Function

Definitions

for a discrete RV \mathbf{X} we have instead

- **Probability Mass Function (PMF):** $p(\mathbf{x}) \triangleq P_{\mathbf{X}}(\mathbf{X} = \mathbf{x})$
- in the above properties we can remove $d\mathbf{x}$ and replace integrals with sums
- the CDF can be defined as

$$F(\mathbf{x}) \triangleq P_{\mathbf{X}}(\mathbf{X} \leq \mathbf{x}) = \sum_{\xi_i \leq \mathbf{x}} p(\xi_i)$$

Joint PDF

Some Properties

reconsider

$$\textcircled{1} F(\mathbf{x}) = P_{\mathbf{X}}(\mathbf{X} \leq \mathbf{x}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_D} p(\xi_1, \dots, \xi_D) d\xi_1 \dots d\xi_D$$

$$\textcircled{2} P_{\mathbf{X}}(\mathbf{x} < \mathbf{X} \leq \mathbf{x} + d\mathbf{x}) \approx p(\mathbf{x}) dx_1 \dots dx_D = p(\mathbf{x}) d\mathbf{x}$$

- the first implies $\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p(\mathbf{x}) d\mathbf{x} = 1$ (consider $(x_1, \dots, x_D) \rightarrow (\infty, \dots, \infty)$))
- the second implies $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^D$
- it is possible that $p(\mathbf{x}) > 1$ for some $\mathbf{x} \in \mathbb{R}^D$

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Joint PDF

Marginal PDF

suppose we want the PDF of $\mathbf{Q} \triangleq (X_1, X_2, \dots, X_{D-1})^T$ (we don't care about X_D)

- $F_{\mathbf{Q}}(\mathbf{q}) = P_{\mathbf{Q}}(\mathbf{Q} \leq \mathbf{q}) = P_{\mathbf{X}}(\mathbf{X} \leq (\mathbf{q}, \infty)^T)$ (X_D can take any value in $(-\infty, \infty)$)
- $P_{\mathbf{X}}(\mathbf{X} \leq (\mathbf{q}, \infty)) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_{D-1}} \int_{-\infty}^{\infty} p_{\mathbf{X}}(\mathbf{x}) dx_1 \dots dx_{D-1} dx_D = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_{D-1}} \left(\int_{-\infty}^{\infty} p_{\mathbf{X}}(\mathbf{x}) dx_D \right) dx_1 \dots dx_{D-1}$
- hence we can define the **marginal PDF**

$$p_{\mathbf{Q}}(\mathbf{q}) = p_{\mathbf{Q}}(x_1, \dots, x_{D-1}) \triangleq \int_{-\infty}^{\infty} p_{\mathbf{X}}(x_1, \dots, x_D) dx_D$$

and one has

$$F_{\mathbf{Q}}(\mathbf{q}) = F_{\mathbf{Q}}(x_1, \dots, x_{D-1}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_{D-1}} p_{\mathbf{Q}}(\mathbf{q}) dx_1 \dots dx_{D-1}$$

- the above procedure can be also used to marginalize more variables
- the above procedure can be used for obtaining a **marginal PMF** for discrete variables by removing the $d\mathbf{x}$ and replacing integrals with sums, i.e.

$$p_{\mathbf{Q}}(\mathbf{q}) = p_{\mathbf{Q}}(x_1, x_2, \dots, x_{D-1}) \triangleq \sum_{x_D} p_{\mathbf{X}}(x_1, x_2, \dots, x_D)$$

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Conditional PDF and Independence

suppose we want the PDF of $\mathbf{Q} \triangleq (X_1, X_2, \dots, X_{D-1})^T$ given $X_D = x_D$

- $P_{\mathbf{Q}|X_D}(\mathbf{q} < \mathbf{Q} \leq \mathbf{q} + d\mathbf{q} \mid x_D < X_D \leq x_D + dx_D) = \frac{P_{\mathbf{X}}(\mathbf{x} < \mathbf{X} \leq \mathbf{x} + d\mathbf{x})}{P_{X_D}(x_D < X_D \leq x_D + dx_D)}$
- $P_{\mathbf{X}}(\mathbf{x} < \mathbf{X} \leq \mathbf{x} + d\mathbf{x}) \approx p_{\mathbf{X}}(\mathbf{x})dx_1 \dots dx_D$
- $P_{X_D}(x_D < X_D \leq x_D + dx_D) \approx p_{X_D}(x_D)dx_D$
- hence $P_{\mathbf{Q}|X_D}(\mathbf{q} < \mathbf{Q} \leq \mathbf{q} + d\mathbf{q} \mid x_D < X_D \leq x_D + dx_D) \approx \frac{p_{\mathbf{X}}(\mathbf{x})}{p_{X_D}(x_D)} dx_1 \dots dx_{D-1}$
- we can define the **conditional PDF**

$$p_{\mathbf{Q}|X_D}(\mathbf{q}|x_D) = p_{\mathbf{Q}|X_D}(x_1, \dots, x_{D-1}|x_D) \triangleq \frac{p_{\mathbf{X}}(\mathbf{x})}{p_{X_D}(x_D)}$$

- \mathbf{Q} and X_D are **independent** $\iff p_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{Q}}(\mathbf{q})p_{X_D}(x_D)$
- if \mathbf{Q} and X_D are independent then $p_{\mathbf{Q}|X_D}(\mathbf{q}|x_D) = p_{\mathbf{Q}}(\mathbf{q})$
- the above definitions can be generalized to define the conditional PDF w.r.t. more variables

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Covariance

- **covariance** of two RVs X and Y

$$\text{cov}[X, Y] \triangleq \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \quad (= \text{cov}[Y, X])$$

- if $\mathbf{x} \in \mathbb{R}^d$, the **mean value** is

$$\mathbb{E}[\mathbf{x}] \triangleq \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{x} p(\mathbf{x}) d\mathbf{x} = \begin{bmatrix} \mathbb{E}[x_1] \\ \vdots \\ \mathbb{E}[x_D] \end{bmatrix} \in \mathbb{R}^D$$

- if $\mathbf{x} \in \mathbb{R}^d$, the **covariance matrix** is

$$\begin{aligned} \text{cov}[\mathbf{x}] &\triangleq \mathbb{E}[(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^T] = \\ &= \begin{bmatrix} \text{var}[X_1] & \text{cov}[X_1, X_2] & \dots & \text{cov}[X_1, X_d] \\ \text{cov}[X_2, X_1] & \text{var}[X_2] & \dots & \text{cov}[X_2, X_d] \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}[X_d, X_1] & \text{cov}[X_d, X_2] & \dots & \text{var}[X_d] \end{bmatrix} \in \mathbb{R}^{D \times D} \end{aligned}$$

- N.B.: $\text{cov}[\mathbf{x}] = \text{cov}[\mathbf{x}]^T$ and $\text{cov}[\mathbf{x}] \geq 0$

- **correlation coefficient** of two RVs X and Y

$$\text{corr}[X, Y] \triangleq \frac{\text{cov}[X, Y]}{\sqrt{\text{var}[X] \text{var}[Y]}}$$

it can be shown that $-1 \leq \text{corr}[X, Y] \leq 1$ (homework¹)

- $\text{corr}[X, Y] = 0 \iff \text{cov}[X, Y] = 0$
- if $\mathbf{x} \in \mathbb{R}^d$, its **correlation** matrix is

$$\text{corr}[\mathbf{x}] \triangleq \begin{bmatrix} \text{corr}[X_1, X_1] & \text{corr}[X_1, X_2] & \dots & \text{corr}[X_1, X_d] \\ \text{corr}[X_2, X_1] & \text{corr}[X_2, X_2] & \dots & \text{corr}[X_2, X_d] \\ \vdots & \vdots & \ddots & \vdots \\ \text{corr}[X_d, X_1] & \text{corr}[X_d, X_2] & \dots & \text{corr}[X_d, X_d] \end{bmatrix} \in \mathbb{R}^{D \times D}$$

- N.B.: $\text{corr}[\mathbf{x}] = \text{corr}[\mathbf{x}]^T$

¹use the fact that $(\int f(t)g(t)dt)^2 \leq \int f^2 dt \int g^2 dt$

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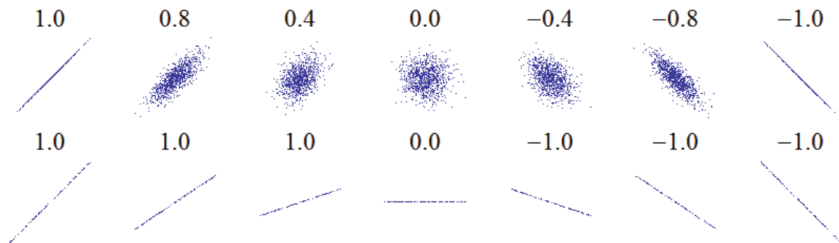
- Joint Probability Distributions
- Joint CDF and PDF
- Marginal PDF
- Conditional PDF and Independence
- Covariance and Correlation
- **Correlation and Independence**
- Common Multivariate Distributions

Correlation and Independence

- **Property 1:** there is a linear relationship between X and Y iff $\text{corr}[X, Y] = 1$, i.e.

$$\text{corr}[X, Y] = 1 \iff Y = aX + b$$

- the correlation coefficient represents a degree of **linear relationship**



Correlation and Independence

- **Property 2:** if X and Y are independent ($p(X, Y) = p(X)p(Y)$) then $\text{cov}[X, Y] = 0$ and $\text{corr}[X, Y] = 0$, i.e. (homework)

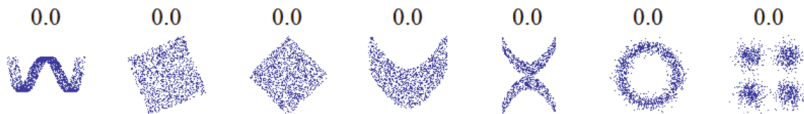
$$X \perp Y \implies \text{corr}[X, Y] = 0$$

- **Property 3:**

$$\text{corr}[X, Y] = 0 \not\Rightarrow X \perp Y$$

example: with $X \sim U(-1, 1)$ and $Y = X^2$ (quadratic dependency) one has $\text{corr}[X, Y] = 0$ (homework)

- other examples where $\text{corr}[X, Y] = 0$ but there is a clear dependence between X and Y



- a more general measure of dependence is the mutual information

1 Common Continuous Distributions - Univariate

- Gaussian Distribution
- Degenerate PDFs
- Student's t Distribution
- Laplace Distribution
- Gamma Distribution
- Beta Distribution
- Pareto Distribution

2 Joint Probability Distributions - Multivariate

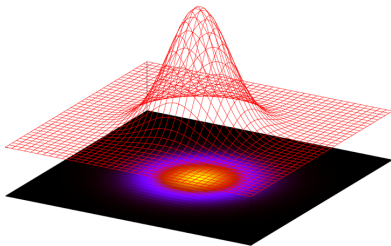
- Joint Probability Distributions
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Multivariate Gaussian (Normal) Distribution

- \mathbf{X} is a continuous RV with values $\mathbf{x} \in \mathbb{R}^D$
- $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, i.e. \mathbf{X} has a **Multivariate Normal** distribution (MVN) or **multivariate Gaussian**

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \triangleq \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

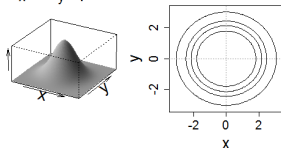
- mean: $\mathbb{E}[\mathbf{x}] = \boldsymbol{\mu}$
- mode: $\boldsymbol{\mu}$
- covariance matrix: $\text{cov}[\mathbf{x}] = \boldsymbol{\Sigma} \in \mathbb{R}^{D \times D}$ where $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T$ and $\boldsymbol{\Sigma} \geq 0$
- precision matrix: $\boldsymbol{\Lambda} \triangleq \boldsymbol{\Sigma}^{-1}$
- spherical isotropic covariance with $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_D$



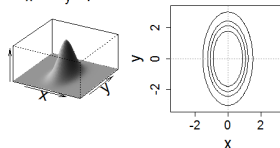
Multivariate Gaussian (Normal) Distribution

Bivariate Normal

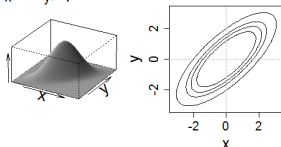
$$\sigma_x = \sigma_y, \rho = 0$$



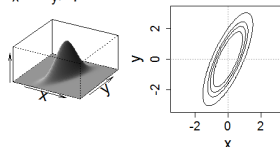
$$2\sigma_x = \sigma_y, \rho = 0$$



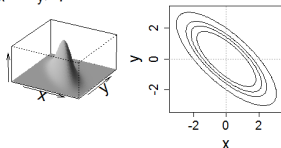
$$\sigma_x = \sigma_y, \rho = 0.75$$



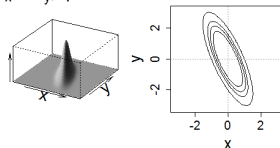
$$2\sigma_x = \sigma_y, \rho = 0.75$$



$$\sigma_x = \sigma_y, \rho = -0.75$$



$$2\sigma_x = \sigma_y, \rho = -0.75$$



Multivariate Student t Distribution

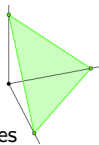
- \mathbf{X} is a continuous RV with values $\mathbf{x} \in \mathbb{R}^D$
- $\mathbf{X} \sim \mathcal{T}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$, i.e. \mathbf{X} has a **Multivariate Student t** distribution

$$\mathcal{T}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu) \triangleq \frac{\Gamma(\nu/2 + D/2)}{\Gamma(\nu/2)} \frac{|\boldsymbol{\Sigma}|^{-1/2}}{\nu^{D/2} \pi^{D/2}} \left[1 + \frac{1}{\nu} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]^{-\left(\frac{\nu+D}{2}\right)}$$

- mean: $\mathbb{E}[\mathbf{x}] = \boldsymbol{\mu}$
- mode: $\boldsymbol{\mu}$
- $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T$ is now called the scale matrix
- covariance matrix: $\text{cov}[\mathbf{x}] = \frac{\nu}{\nu-2} \boldsymbol{\Sigma}$
- N.B.: this distribution is similar to MVN but it's **more robust** w.r.t outliers due to its fatter tails (see the previous slides about univariate Student t distribution)

Dirichlet Distribution

- \mathbf{X} is a continuous RV with values $\mathbf{x} \in S_K$
- **probability simplex** $S_K \triangleq \{\mathbf{x} \in \mathbb{R}^K : 0 \leq x_i \leq 1, \sum_{i=1}^K x_i = 1\}$
- the vector $\mathbf{x} = (x_1, \dots, x_K)$ can be used to represent a set of K probabilities
- $\mathbf{X} \sim \text{Dir}(\alpha)$, i.e. \mathbf{X} has a **Dirichlet** distribution



$$\text{Dir}(\mathbf{x}|\alpha) \triangleq \frac{1}{B(\alpha)} \prod_{i=1}^K x_i^{\alpha_i-1} \mathbb{I}(\mathbf{x} \in S_K)$$

where $\alpha \in \mathbb{R}^K$ and $B(\alpha)$ is a generalization of the beta function to K variables²

$$B(\alpha) = B(\alpha_1, \dots, \alpha_K) \triangleq \frac{\prod_{i=1}^K \Gamma(\alpha_i)}{\Gamma(\alpha_0)}$$

- $\alpha_0 = \sum_{i=1}^K \alpha_i$
- $\mathbb{E}[x_k] = \frac{\alpha_k}{\alpha_0}$, $\text{mode}[x_k] = \frac{\alpha_k - 1}{\alpha_0 - K}$, $\text{var}[x_k] = \frac{\alpha_k(\alpha_0 - \alpha_k)}{\alpha_0^2(\alpha_0 + 1)}$
- N.B.: this distribution is a multivariate generalization of the beta distribution

²see the slide about the gamma distribution for the definition of $\Gamma(\alpha)$

- Kevin Murphy's book